Constructing a Family of 4-Critical Planar Graphs with High Edge-Density

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Abstract

A graph $G = (V, E)$ is a k-critical graph if G is not $(k-1)$-colorable but $G - e$ is $(k-1)$-colorable for every $e \in E(G)$. In this paper, we construct a family of 4-critical planar graphs with n vertices and $\frac{7n-13}{3}$ edges. As a consequence, this improved the bound for the maximum edge density obtained by Abbott and Zhou. We conjecture that this is the largest edge density for a 4-critical planar graph.

1 Introduction

Let $G = (V, E)$ be a graph, G is said to be k-colorable if there is a assignment of k colors to the vertices of G such that no two adjacent vertices of G get the same color. The chromatic number of G, denoted by $\chi(G)$, is the least integer k such that G is k-colorable. A graph $G = (V, E)$ is a k-critical graph if G is not $(k-1)$-colorable but $G - e$ is $(k-1)$-colorable for every $e \in E(G)$. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Let $G = (V, E)$ be a graph and $v \in V(G)$, we denote by $N(v)$ the set of vertices that are adjacent to v. Let $f(n)$ and $F(n)$ denote respectively the minimum number and maximum number of a 4-critical planar graph with n vertices. Table 1 shows some exact values of $f(n)$ and $F(n)$ for $n \leq 14$. In [5], A.V. Kostochka and M. Yancey proved that $f(n) \geq \frac{5n-2}{3}$, and this bound is sharp in the sense that there are infinitely many 4-critical planar graphs on n vertices and $\frac{5n-2}{3}$ edges. As for $F(n)$, H. L. Abbott and B. Zhou [1] proved that $F(n) \leq 2.75n$, G. Koster [2] later improved this bound to $\frac{5n}{2}$. We believe that this upper bound can not be obtained. Let $G = (V, E)$ be a graph, define $S = \sup|E(G)|/|V(G)|$, where the bounds are taken over all 4-critical planar graphs with $|V(G)|$ vertices and $|E(G)|$ edges. Grünbaum [3] used Hajós's construction [4] to show that $S \geq 79/39 = 2.02564...$; and he asked the question of determining the maximum edge density of planar 4-critical graphs. Abbott and Zhou [1] used a variation of the Hajós's construction to

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show that $S \geq 39/19 = 2.05263....$ In this paper, we prove that $S \geq \frac{7}{3}$ by constructing a family of 4-critical planar graphs on n vertices and $\frac{7n-13}{3}$ edges.

n	6	7	8	9	10	11	12	13	14
$f(n)$	10	11	14	15	16	19	20	21	24
$F(n)$	10	12	14	16	18	20	22	26	28

Table 1: Some values of $f(n)$ and $F(n)$

2 Main Results

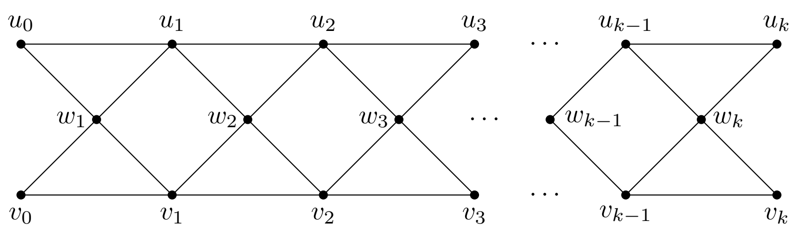


Figure 1: The graph H_k

Lemma 2.1. *Let H_k be the graph shown in Figure 1, then H_k is three colorable; Moreover, let $c : V(H_k) \to \{1, 2, 3\}$ be a 3-coloring of H_k,*

(i) if there is nonnegative integer i $(0 \leq i \leq k)$ such that $c(u_i) = c(v_i)$, then $c(u_j) = c(v_j)$ for all $j = 0, 1, 2, \cdots, k$;

(ii) if there is nonnegative integer i $(0 \leq i \leq k)$ such that $c(u_i) \neq c(v_i)$, then each w_i gets the same third color, and $c(u_j) \neq c(v_j)$ for all $j = 0, 1, 2, \cdots, k$;

Proof. We color w_i $(1 \leq i \leq k)$ the color 1, and for each pair of vertices u_{2i} and v_{2i}, we color them the color 2; finally, all the remaining vertices are colored 3. It is easy to see that it is a 3-coloring of H_k.

(i) It is easy to check that (i) is valid for H_1; Assume that (i) holds for H_{k-1}. Now consider H_k, suppose there is nonnegative integer i $(0 \leq i \leq k)$ such that $c(u_i) = c(v_i)$ (say $c(u_0) = c(v_0)$). Note that $H_{k-1} = H_k - \{u_k, v_k, w_k\}$, by assumption, we have $c(u_j) = c(v_j)$ for all $j = 0, 1, 2, \cdots, k-1$; without loss of generality, assume that $c(u_{k-1}) = c(v_{k-1}) = 1$. Now, if $w_k = 2$, then $c(u_{k-1}) = c(v_{k-1}) = 3$; and if $w_k = 3$, then $c(u_{k-1}) = c(v_{k-1}) = 2$; This proves (i) by induction.

(ii) We can prove it also by induction, the details are omitted here. ■

Theorem 2.2. *For each positive integer k, there is a 4-critical planar graph with $6k+7$ vertices and $14k + 12$ edges.*

Corollary 2.3. $S \geq \frac{7}{3}$.

Proof of theorem 2.2. We construct a graph G_k as shown in Figure 2, where $V(G_k) = V(H_{2k}) \cup \{x_1, x_2, x_3, y_1, y_2\}$ and

$$\begin{aligned} E(G_k) = E(H_{2k}) \quad & \cup \quad \{x_1u_0, x_1u_2, \cdots, x_1u_{2k}\} \\ & \cup \quad \{y_1v_1, y_1v_3, \cdots, y_1v_{2k-1}\} \\ & \cup \quad \{x_1x_2, x_1x_3, x_1y_2, x_2y_1, x_2y_2, x_2v_0, x_2x_3\} \\ & \cup \quad \{x_3y_1, x_3v_{2k}, x_3u_{2k}, y_2w_1\} \end{aligned}$$

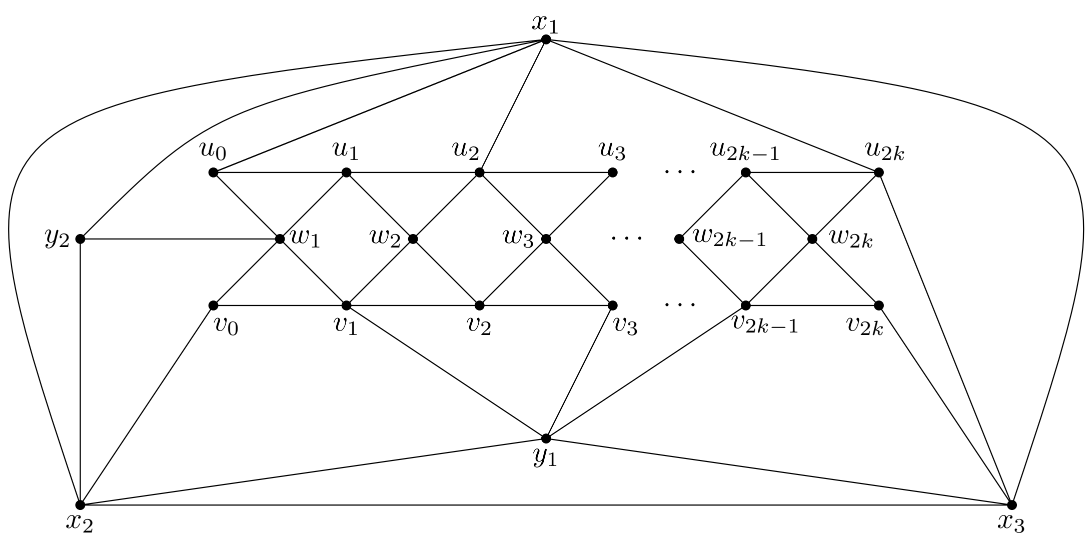
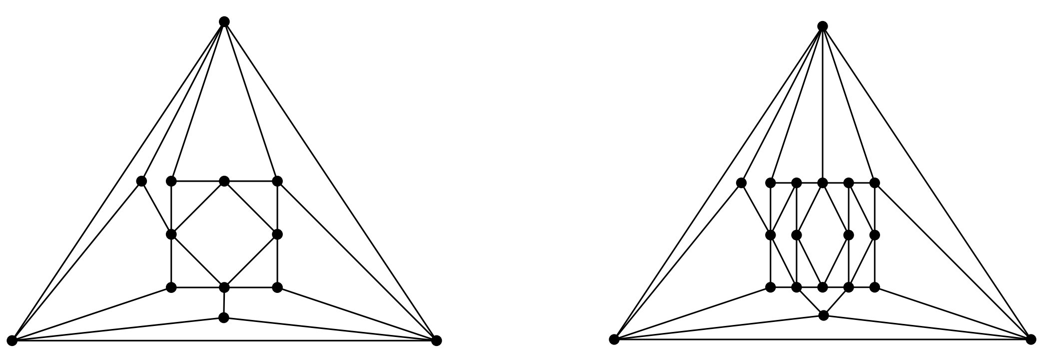


Figure 2: The graph G_k

When $k = 1$ and $k = 2$, see Figure 3 in particular, we can check that both of them are 4-critical planar graphs.



(a) G_1 (b) G_2

Figure 3: The 4-critical graphs G_1 and G_2

Note that G_k is a planar graph with $6k+7$ vertices and $14k+12$ edges. In the following, we shall prove that both G_k is 4-critical.

First, we prove that G_k is not 3-colorable. Suppose that G_k is 3-colorable, let $c : V(G_k) \to \{1, 2, 3\}$ be a 3-coloring of G_k. Note that $\{x_1, x_2, x_3\}$ form a triangle in G_k, without loss of generality, assume that $c(x_1) = 1, c(x_2) = 2, c(x_3) = 3$, then $c(y_1) = 1, c(y_2) = 3, c(u_{2k}) = 2$. Note that $c(v_{2k}) \in \{1, 2\}$ since x_3 and v_{2k} are adjacent. If $c(v_{2k}) = c(u_{2k}) = 2$, then $c(v_{2k-1}) = 3$ since v_{2k-1} is adjacent to both y_1 and v_{2k}. By Lemma 2.1, we have that $c(u_{2k-1}) = c(v_{2k-1}) = 3$. By this way, we get in general that for each $u_i \in N(x_1)$, $c(u_i) = c(v_i) = 2$; and for each $v_j \in N(y_1)$, $c(v_j) = c(u_j) = 3$. So $c(v_0) = 2$, this is impossible since $c(x_2) = 2$. If $c(v_{2k}) \neq c(u_{2k})$, then $c(v_{2k}) = 1$. By Lemma 2.1, $c(w_i) = 3$ for $1 \leq i \leq 2k$. So $c(y_2) = c(w_1) = 3$, this is impossible since y_2 and w_1 are adjacent. Therefore, G_k is not 3-colorable.

Next, we prove that for each $e \in E(G_k)$, $G_k - e$ is 3-colorable. Since the number of automorphisms of G_k is one, G_k is extremely not symmetric, so we have to consider awkwardly every possible edge of G_k. In the following, let

$$U_1 = N(x_1) \cap V(H_{2k});$$
$$U_2 = \{u_0, u_1, u_2, \cdots, u_{2k}\} - U_1;$$
$$W = \{w_1, w_2, \cdots, w_{2k}\};$$
$$V_1 = N(y_1) \cap V(H_{2k});$$
$$V_2 = \{v_0, v_1, v_2, \cdots, v_{2k}\} - V_1;$$
$$[u_i, u_j] = \{u_i, u_{i+1}, u_{i+2}, \cdots, u_j\}, \text{ where } i < j;$$
$$[u_i : u_j] = \{u_t \mid u_t \in [u_i, u_j] \text{ and } t - i \text{ is even}\}, \text{ where } i < j.$$

Furthermore, if a graph G is 3-colorable, we will denote by C_1 and C_2 the set of vertices that are colored 1 and 2 respectively. For the sake of conciseness, we will not indicate the set of vertices that are colored 3.

Note that G_k has four vertices y_2, u_0, v_0, v_{2k} which have degree 3. If $G_k - v$ is 3-colorable for $v \in \{y_2, u_0, v_0, v_{2k}\}$, then it is obvious that $G_k - e$ is 3-colorable for each edge e that is incident with v. Let $H = G_k - v$ for some $v \in \{y_2, u_0, v_0, v_{2k}\}$, we first prove that H is 3-colorable.

If $H = G_k - y_2$, let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_3\} \cup W$;

If $H = G_k - u_0$, let $C_1 = \{x_1, y_1, v_0\} \cup [w_2, w_{2k}]$, $C_2 = \{x_3, y_2\} \cup U_2 \cup V_1$;

If $H = G_k - v_0$, let $C_1 = \{x_1, y_1\} \cup W$, $C_2 = \{x_3, y_2\} \cup U_2 \cup V_1$;

If $H = G_k - v_{2k}$, let $C_1 = \{x_1, y_1\} \cup W$, $C_2 = \{x_2\} \cup U_1 \cup V_1$.

This proves that $H = G_k - v$ is 3-colorable for some $v \in \{y_2, u_0, v_0, v_{2k}\}$.

Next, let e be an edge that is not incident with any vertex of $\{y_2, u_0, v_0, v_{2k}\}$ and let $G = G_k - e$, we shall prove that G is 3-colorable.

If $e = x_1x_2$, let $C_1 = \{x_1, x_2\} \cup W$, $C_2 = \{y_1\} \cup U_1 \cup V_2$;

If $e = x_1x_3$, let $C_1 = \{x_1, x_3\} \cup W$, $C_2 = \{y_1, y_2\} \cup U_1 \cup V_2$;

If $e = x_2x_3$, let $C_1 = \{x_1\} \cup W$, $C_2 = \{y_1, y_2\} \cup U_1 \cup V_2$;

If $e = x_2y_1$, let $C_1 = \{x_1\} \cup U_2 \cup V_1$, $C_2 = \{x_3, y_2, u_0, v_0\} \cup [w_2, w_{2k}]$;

If $e = x_3y_1$, let $C_1 = \{x_1\} \cup U_2 \cup V_1$, $C_2 = \{x_3, y_1, y_2, u_0, v_0\} \cup [w_2, w_{2k}]$;

If $e = x_3u_{2k}$, let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_2\} \cup W$;

If $e = x_1u_i$ (i is even), let $C_1 = \{x_1, y_1, u_i, v_i\} \cup W - \{w_i, w_{i+1}\}$, $C_2 = \{x_3, y_2, w_i\} \cup [u_0 : u_{i-2}] \cup [u_{i+1} : u_{2k-1}] \cup [v_0 : v_{i-2}] \cup [v_{i+1} : v_{2k-1}]$;

If $e = y_1v_i$ (i is odd), let $C_1 = \{x_1, y_1, u_i, v_i\} \cup W - \{w_i, w_{i+1}\}$, $C_2 = \{x_2, w_i\} \cup [u_1 : u_{i-2}] \cup [u_{i+1} : u_{2k}] \cup [v_1 : v_{i-2}] \cup [v_{i+1} : v_{2k}]$;

If $e = u_iu_{i+1}$ (i is odd), let $C_1 = \{x_1, y_1\} \cup [u_{i+2} : u_{2k-1}] \cup [v_{i+1} : v_{2k}] \cup [w_1, w_i]$, $C_2 = \{x_2\} \cup [u_1 : u_i] \cup [u_{i+1} : u_{2k}] \cup V_1$;

If $e = u_iu_{i+1}$ (i is even), let $C_1 = \{x_1, y_1\} \cup [u_1 : u_{i-1}] \cup [v_0 : v_i] \cup [w_{i+2}, w_{2k}]$, $C_2 = \{x_3, y_2\} \cup [u_0 : u_i] \cup [u_{i+1} : u_{2k-1}] \cup V_1$;

If $e = v_iv_{i+1}$ (i is even), let $C_1 = \{x_1, y_1\} \cup W$, $C_2 = \{x_3, y_2\} \cup [v_0 : v_i] \cup [v_{i+1} : v_{2k-1}] \cup U_2$;

If $e = v_iv_{i+1}$ (i is odd), let $C_1 = \{x_1, y_1\} \cup W$, $C_2 = \{x_3, y_2\} \cup [v_0 : v_{i-1}] \cup [v_{i+2} : v_{2k-1}] \cup U_2$;

If $e = u_iw_i$ ($u_i \in U_2$), let $C_1 = \{x_1, y_1\} \cup [u_i : u_{2k-1}] \cup [v_{i+1} : v_{2k}] \cup [w_1, w_i]$, $C_2 = \{x_2\} \cup U_1 \cup V_1$;

If $e = u_iw_{i+1}$ ($u_i \in U_2$), let $C_1 = \{x_1, y_1\} \cup [u_{i+2} : u_{2k-1}] \cup [v_{i+1} : v_{2k}] \cup [w_1, w_i]$, $C_2 = \{x_2\} \cup U_1 \cup V_1$;

If $e = u_iw_{i+1}$ ($u_i \in U_1$), let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_3, y_2\} \cup [u_0 : u_i] \cup [v_1 : v_{i-1}] \cup [w_{i+1}, w_{2k}]$;

If $e = u_iw_i$ ($u_i \in U_1$), let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_3, y_2\} \cup [u_0 : u_{i-1}] \cup [v_1 : v_{i-1}] \cup [w_{i+1}, w_{2k}]$;

If $e = v_{i+1}w_{i+1}$ ($v_{i+1} \in V_1$), let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_3, y_2\} \cup [u_0 : u_{i-1}] \cup [v_1 : v_{i-1}] \cup [w_{i+3}, w_{2k}]$;

If $e = v_{i-1}w_i$ ($v_{i-1} \in V_1$), let $C_1 = \{x_1, y_1\} \cup U_2 \cup V_2$, $C_2 = \{x_3, y_2\} \cup [u_0 : u_{i-1}] \cup [v_1 : v_{i-1}] \cup [w_i, w_{2k}]$;

If $e = v_iw_i$ ($v_i \in V_2$), let $C_1 = \{x_1, y_1\} \cup [u_{i+1} : u_{2k-1}] \cup [v_i : v_{2k}] \cup [w_1, w_i]$, $C_2 = \{x_2\} \cup U_1 \cup V_1$;

If $e = v_iw_{i+1}$ ($v_i \in V_2$), let $C_1 = \{x_1, y_1\} \cup [u_{i+1} : u_{2k-1}] \cup [v_{i+2} : v_{2k}] \cup [w_1, w_i]$, $C_2 = \{x_2\} \cup U_1 \cup V_1$.

From the above coloring schedules, we have checked that for each $e \in E(G_k)$, $G_k - e$ is 3-colorable. This completes the proof of theorem 2.2. ■

3 Some remarks and problems

- Note that both G_k and G_k^2 have minimum degree 3. Is there a 4-critical planar graph on $6k + 7$ vertices and $14k + 12$ edges and with $\delta \geq 4$?

- Note that in Table 1, all the values of $F(n)$ for small n are even. Is it true that $F(n)$ is even for all positive integer n?
- We conjecture that $S = \frac{7}{3}$. More concisely, we conjecture that $F(n) \leq \frac{7n-13}{3}$, where the equality holds only if $n \equiv 1 (\mathrm{mod}\ 6)$.

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